

PROJECTIVE DIMENSIONS AND EXTENSIONS OF MODULES FROM TILTED TO CLUSTER-TILTED ALGEBRAS

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Abstract

We study the module categories of a tilted algebra C and the corresponding cluster-tilted algebra $B = C \ltimes E$ where E is the C - C -bimodule $\text{Ext}_C^2(DC, C)$. We investigate how various properties of a C -module are affected when considered in the module category of B . We give a complete classification of the projective dimension of a C -module inside $\text{mod } B$. If a C -module M satisfies $\text{Ext}_C^1(M, M) = 0$, we show two sufficient conditions for M to satisfy $\text{Ext}_B^1(M, M) = 0$. In particular, if M is indecomposable and $\text{Ext}_C^1(M, M) = 0$, we prove M always satisfies $\text{Ext}_B^1(M, M) = 0$.

1 Introduction

We are interested in studying the representation theory of cluster-tilted algebras which are finite dimensional associative algebras that were introduced by Buan, Marsh, and Reiten in [13] and, independently, by Caldero, Chapoton, and Schiffler in [16] for type \mathbb{A} .

One motivation for introducing these algebras came from Fomin and Zelevinsky's cluster algebras [18]. Cluster algebras were developed as a tool to study dual canonical bases and total positivity in semisimple Lie groups, and cluster-tilted algebras were constructed as a categorification of these algebras. To every cluster in an acyclic cluster algebra one can associate a cluster-tilted algebra, such that the indecomposable rigid modules over the cluster-tilted algebra correspond bijectively to the cluster variables outside the chosen cluster. Many people have studied cluster-tilted algebras in this context, see for example [10, 13, 14, 15, 17, 20].

The second motivation came from classical tilting theory. Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, whereas cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster

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categories of hereditary algebras. This similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established in [2].

There is a surjective map

$$\{\text{tilted algebras}\} \mapsto \{\text{cluster-tilted algebras}\}$$

$$C \mapsto B = C \ltimes E$$

where E denotes the C - C -bimodule $E = \text{Ext}_C^2(DC, C)$ and $C \ltimes E$ is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. It is natural to ask how the module categories of C and B are related and several results in this direction have been obtained, see for example [3, 4, 5, 9, 11]. In this work, we investigate how various properties of a C -module are affected when the same module is viewed as a B -module via the standard embedding. We let M be a right C -module and define a right $B = C \ltimes E$ action on M by

$$M \times B \rightarrow M, \quad (m, (c, e)) \mapsto mc.$$

Our first main result is on the projective dimension of a C -module when viewed as a B -module. Here, τ_C^{-1} and Ω_C^{-1} denote respectively the inverse Auslander-Reiten translation and first cosyzygy of a C -module.

Theorem 1.1. *Let C be a tilted algebra, $E = \text{Ext}_C^2(DC, C)$, and $B = C \ltimes E$ the corresponding cluster-tilted algebra.*

- (a) *If $\text{pd}_C M = 0$, then $\text{pd}_B M = 0$ if and only if $\text{id}_C M \leq 1$. Otherwise, $\text{pd}_B M = \infty$.*
- (b) *If $\text{pd}_C M = 2$, then $\text{pd}_B M = \infty$.*
- (c) *Let $\text{pd}_C M = 1$ with minimal projective resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Then $\text{pd}_B M = 1$ if and only if $\text{id}_C M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_0 \cong \tau_C^{-1}\Omega_C^{-1}P_1$. Otherwise, $\text{pd}_B M = \infty$.*

Our second main result is on C -modules that satisfy $\text{Ext}_C^1(M, M) = 0$. These are known as *rigid* modules. Here, our result holds in a more general setting with C an algebra of global dimension equal to 2. We determine two sufficient conditions to guarantee when a rigid C -module remains rigid when viewed as a B -module, i.e., $\text{Ext}_B^1(M, M) = 0$. Here, τ_C and Ω_C denote respectively the Auslander-Reiten translation and first syzygy of a C -module.

Theorem 1.2. *Let M be a rigid C -module with a projective cover $P_0 \rightarrow M$ and an injective envelope $M \rightarrow I_0$ in $\text{mod } C$.*

- (a) *If $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$, then M is a rigid B -module.*
- (b) *If $\text{Hom}_C(M, \tau_C\Omega_C I_0) = 0$, then M is a rigid B -module.*

As an immediate consequence, in the case C is tilted, we obtain an affirmative answer to whether an indecomposable rigid C -module remains rigid as a B -module.

Corollary 1.3. *Let C be a tilted algebra with B the corresponding cluster-tilted algebra. Suppose M is an indecomposable, rigid C -module. Then M is a rigid B -module.*

2 Notation and Preliminaries

We now set the notation for the remainder of this paper. All algebras are assumed to be finite dimensional over an algebraically closed field k . Suppose $Q = (Q_0, Q_1)$ is a connected quiver without oriented cycles where Q_0 denotes a finite set of vertices and Q_1 denotes a finite set of oriented arrows. By kQ we denote the path algebra of Q . If Λ is a k -algebra then denote by $\text{mod } \Lambda$ the category of finitely generated right Λ -modules and by $\text{ind } \Lambda$ a set of representatives of each isomorphism class of indecomposable right Λ -modules. Given $M \in \text{mod } \Lambda$, the projective dimension of M in $\text{mod } \Lambda$ is denoted $\text{pd}_\Lambda M$ and its injective dimension by $\text{id}_\Lambda M$. We denote by $\text{add } M$ the smallest additive full subcategory of $\text{mod } \Lambda$ containing M , that is, the full subcategory of $\text{mod } \Lambda$ whose objects are the direct sums of direct summands of the module M . As mentioned before, we let τ_Λ and τ_Λ^{-1} be the Auslander-Reiten translations in $\text{mod } \Lambda$. We let D be the standard duality functor $\text{Hom}_k(-, k)$. Also mentioned before, ΩM and $\Omega^{-1} M$ will denote the first syzygy and first cosyzygy of M . Finally, let gl.dim stand for the global dimension of an algebra.

2.1 Tilted Algebras

Tilting theory is one of the main themes in the study of the representation theory of algebras. Given a k -algebra A , one can construct a new algebra B in such a way that the corresponding module categories are closely related. The main idea is that of a tilting module.

Definition 2.1. Let A be an algebra. An A -module T is a *partial tilting module* if the following two conditions are satisfied:

- (1) $\text{pd}_A T \leq 1$.
- (2) $\text{Ext}_A^1(T, T) = 0$.

A partial tilting module T is called a *tilting module* if it also satisfies the following additional condition:

- (3) There exists a short exact sequence $0 \rightarrow A \rightarrow T' \rightarrow T'' \rightarrow 0$ in $\text{mod } A$ with T' and $T'' \in \text{add } T$.

Partial tilting modules induce torsion pairs in a natural way. We consider the restriction to a subcategory C of a functor F defined originally on a module category, and we denote it by $F|_C$. Also, let S be a subcategory of a category C . We say S is a *full subcategory* of C if, for each pair of objects X and Y of S , $\text{Hom}_S(X, Y) = \text{Hom}_C(X, Y)$.

Definition 2.2. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\text{mod } A$ is called a *torsion pair* if the following conditions are satisfied:

- (a) $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.
- (b) $\text{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
- (c) $\text{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

Consider the following full subcategories of $\text{mod } A$ where T is a partial tilting module.

$$\mathcal{T}(T) = \{M \in \text{mod } A \mid \text{Ext}_A^1(T, M) = 0\}$$

$$\mathcal{F}(T) = \{M \in \text{mod } A \mid \text{Hom}_A(T, M) = 0\}$$

Then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair in $\text{mod } A$ called the *induced torsion pair* of T . Considering the endomorphism algebra $C = \text{End}_A T$, there is an induced torsion pair, $(\mathcal{X}(T), \mathcal{Y}(T))$, in $\text{mod } C$.

$$\mathcal{X}(T) = \{M \in \text{mod } B \mid M \otimes_C T = 0\}$$

$$\mathcal{Y}(T) = \{M \in \text{mod } B \mid \text{Tor}_1^C(M, T) = 0\}$$

We now state the definition of a tilted algebra.

Definition 2.3. Let A be a hereditary algebra with T a tilting A -module. Then the algebra $C = \text{End}_A T$ is called a *tilted algebra*.

The following proposition describes several facts about tilted algebras. Let A be an algebra and M, N be two indecomposable A -modules. A *path* in $\text{mod } A$ from M to N is a sequence

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \dots \xrightarrow{f_s} M_s = N$$

where $s \geq 0$, all the M_i are indecomposable, and all the f_i are nonzero nonisomorphisms. In this case, M is called a *predecessor* of N in $\text{mod } A$ and N is called a *successor* of M in $\text{mod } A$.

Proposition 2.4. [7, VIII, Lemma 3.2.]. *Let A be a hereditary algebra, T a tilting A -module, and $C = \text{End}_A T$ the corresponding tilted algebra. Then*

- (a) $\text{gl.dim } C \leq 2$.
- (b) For all $M \in \text{ind } C$, $\text{id}_C M \leq 1$ or $\text{pd}_C M \leq 1$.
- (c) For all $M \in \mathcal{X}(T)$, $\text{id}_C M \leq 1$.
- (d) For all $M \in \mathcal{Y}(T)$, $\text{pd}_C M \leq 1$.
- (e) $(\mathcal{X}(T), \mathcal{Y}(T))$ is *splitting*, which means that every indecomposable C -module belongs to either $\mathcal{X}(T)$ or $\mathcal{Y}(T)$.
- (f) $\mathcal{Y}(T)$ is closed under predecessors and $\mathcal{X}(T)$ is closed under successors.

2.2 Cluster categories and cluster-tilted algebras

Let $A = kQ$ and let $\mathcal{D}^b(\text{mod } A)$ denote the derived category of bounded complexes of A -modules as summarized in [12]. The *cluster category* C_A is defined as the orbit category of the derived category with respect to the functor $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in the derived category and $[1]$ is the shift. Cluster categories were introduced in [12], and in [16] for type \mathbb{A} , and were further studied in

[1, 19, 20, 21]. They are triangulated categories [19], that are 2-Calabi Yau and have Serre duality [12].

An object T in C_A is called *cluster-tilting* if $\text{Ext}_{C_A}^1(T, T) = 0$ and T has $|Q_0|$ non-isomorphic indecomposable direct summands. The endomorphism algebra $\text{End}_{C_A} T$ of a cluster-tilting object is called a *cluster-tilted algebra* [13].

The following theorem was shown in [20]. It characterizes the homological dimensions of a cluster-tilted algebra.

Theorem 2.5. [20]. *Cluster-tilted algebras are 1-Gorenstein, that is, every projective module has injective dimension at most 1 and every injective module has projective dimension at most 1.*

As an important consequence, the projective dimension and the injective dimension of any module in a cluster-tilted algebra are simultaneously either infinite, or less than or equal to 1 (see [20, Section 2.1]).

2.3 Relation Extensions

Let C be an algebra of global dimension at most 2 and let E be the C - C -bimodule $E = \text{Ext}_C^2(DC, C)$.

Definition 2.6. The *relation extension* of C is the trivial extension $B = C \ltimes E$, whose underlying C -module structure is $C \oplus E$, and multiplication is given by $(c, e)(c', e') = (cc', ce' + ec')$.

Relation extensions were introduced in [2]. In the special case where C is a tilted algebra, we have the following result.

Theorem 2.7. [2]. *Let C be a tilted algebra. Then $B = C \ltimes \text{Ext}_C^2(DC, C)$ is a cluster-tilted algebra. Moreover all cluster-tilted algebras are of this form.*

2.4 Induction and coinduction functors

A fruitful way to study cluster-tilted algebras is via induction and coinduction functors. Recall, D denotes the standard duality functor.

Definition 2.8. Let C be a subalgebra of B such that $1_C = 1_B$, then

$$- \otimes_C B : \text{mod } C \rightarrow \text{mod } B$$

is called the *induction functor*, and dually

$$D(B \otimes_C D-) : \text{mod } C \rightarrow \text{mod } B$$

is called the *coinduction functor*. Moreover, given $M \in \text{mod } C$, the corresponding induced module is defined to be $M \otimes_C B$, and the coinduced module is defined to be $D(B \otimes_C DM)$.

We can say more in the situation when B is a split extension of C . Call a C - C -bimodule E *nilpotent* if, for $n \geq 0$, $E \otimes_C E \otimes_C \cdots \otimes_C E = 0$, where the tensor product is performed n times.

Definition 2.9. Let B and C be two algebras. We say B is a *split extension* of C by a nilpotent bimodule E if there exists a short exact sequence of B -modules

$$0 \rightarrow E \rightarrow B \begin{smallmatrix} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{smallmatrix} C \rightarrow 0$$

where π and σ are algebra morphisms, such that $\pi \circ \sigma = 1_C$, and $E = \ker \pi$ is nilpotent.

In particular, relation extensions are split extensions. The next proposition shows a precise relationship between a given C -module and its image under the induction and coinduction functors.

Proposition 2.10. [22, Proposition 3.6]. *Suppose B is a split extension of C by a nilpotent bi-module E . Then, for every $M \in \text{mod } C$, there exists two short exact sequences of B -modules:*

- (a) $0 \rightarrow M \otimes_C E \rightarrow M \otimes_C B \rightarrow M \rightarrow 0$
- (b) $0 \rightarrow M \rightarrow D(B \otimes_C DM) \rightarrow D(E \otimes_C DM) \rightarrow 0$

The next two results give information on the projective cover and the minimal projective presentation of an induced module.

Lemma 2.11. [6, Lemma 1.3]. *Suppose B is a split extension of C by a nilpotent bimodule E . Let M be a C -module. If $f: P \rightarrow M$ is a projective cover in $\text{mod } C$, then $f \otimes_C 1_B: P \otimes_C B \rightarrow M \otimes_C B$ is a projective cover in $\text{mod } B$.*

Lemma 2.12. [6]. *Suppose B is a split extension of C by a nilpotent bimodule E . Let M be a C -module. If $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ is a projective presentation, then $P_1 \otimes_C B \rightarrow P_0 \otimes_C B \rightarrow M \otimes_C B \rightarrow 0$ is a projective presentation. Furthermore, if the first is minimal, then so is the second.*

The following is a crucial result needed in section 3.

Lemma 2.13. [6, Lemma 2.2]. *For a C -module M , we have $\text{pd}_B(M \otimes_C B) \leq 1$ if and only if $\text{pd}_C M \leq 1$ and $\text{Hom}_C(DE, \tau_C M) = 0$.*

2.5 Standard results

In this subsection we list several standard results which hold over arbitrary k -algebras of finite dimension. We begin with a result on the projective dimension of arbitrary modules related by a short exact sequence.

Lemma 2.14. [7, Appendix, Proposition 4.7]. *Let A be a finite dimensional k -algebra and suppose $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence in $\text{mod } A$.*

- (a) $\text{pd}_A N \leq \max(\text{pd}_A M, 1 + \text{pd}_A L)$, and equality holds if $\text{pd}_A M \neq \text{pd}_A L$.
- (b) $\text{pd}_A L \leq \max(\text{pd}_A M, -1 + \text{pd}_A N)$, and equality holds if $\text{pd}_A M \neq \text{pd}_A N$.
- (c) $\text{pd}_A M \leq \max(\text{pd}_A L, \text{pd}_A N)$, and equality holds if $\text{pd}_A N \neq 1 + \text{pd}_A L$.

The next result, which relates the Ext and Tor functors, will be needed in section 3.

Proposition 2.15. [7, Appendix, Proposition 4.11] *Let A be a finite dimensional k -algebra. For all modules Y and Z in $\text{mod } A$, we have*

$$D\text{Ext}_A^1(Y, DZ) \cong \text{Tor}_1^A(Y, Z).$$

The following lemma is well known.

Lemma 2.16. [7, IV, Lemma 2.7] *Let A be a finite dimensional k -algebra and M an A -module.*

- (a) $\text{pd}_A M \leq 1$ if and only if $\text{Hom}_A(DA, \tau_A M) = 0$
- (b) $\text{id}_A M \leq 1$ if and only if $\text{Hom}_A(\tau_A^{-1} M, A) = 0$

For our next two statements we need two definitions. We say a submodule S of a module M is *superfluous* if, whenever $L \subseteq M$ is a submodule with $L + S = M$, then $L = M$. An epimorphism $f : M \rightarrow N$ is *minimal* if $\ker f$ is superfluous in M . In particular, any projective cover is minimal.

Lemma 2.17. [7, I, Lemma 5.6] *Let A be a finite dimensional k -algebra and M an A -module. Then an epimorphism $f : P \rightarrow M$ is minimal if and only if for any morphism $g : N \rightarrow P$, the surjectivity of $f \circ g$ implies the surjectivity of g .*

Corollary 2.18. *If $g : M \rightarrow N$ and $f : N \rightarrow L$ are epimorphisms and f and g are minimal, then $f \circ g$ is minimal.*

Proof. Clearly, $f \circ g$ is surjective. Thus, we must show that $\ker f \circ g$ is superfluous. Let $h : X \rightarrow M$ be a morphism such that $f \circ g \circ h$ is surjective. Since $f \circ g \circ h = f \circ (g \circ h)$ and f is minimal, we know by Lemma 2.17 that $g \circ h$ is surjective. Since g is minimal, we may use Lemma 2.17 again to say h is surjective. Thus, $f \circ g \circ h$ is surjective and a final application of Lemma 2.17 says that $f \circ g \circ h$ is minimal. \square

2.6 Induced and coinduced modules in cluster-tilted algebras

In this section we cite several properties of the induction and coinduction functors particularly when C is an algebra of global dimension at most 2 and $B = C \ltimes E$ is the trivial extension of C by the C - C -bimodule $E = \text{Ext}_C^2(DC, C)$. In the specific case when C is also a tilted algebra, B is the corresponding cluster-tilted algebra.

Proposition 2.19. [22, Proposition 4.1]. *Let C be an algebra of global dimension at most 2. Then*

- (a) $E \cong \tau_C^{-1} \Omega_C^{-1} C$.
- (b) $DE \cong \tau_C \Omega_C DC$.
- (c) $M \otimes_C E \cong \tau_C^{-1} \Omega_C^{-1} M$.
- (d) $D(E \otimes_C DM) \cong \tau_C \Omega_C M$.

The next two results use homological dimensions to extract information about induced and coinduced modules.

Proposition 2.20. [22, Proposition 4.2]. *Let C be an algebra of global dimension at most 2, and let $B = C \ltimes E$. Suppose $M \in \text{mod } C$, then*

- (a) $\text{id}_C M \leq 1$ if and only if $M \otimes_C B \cong M$.
- (b) $\text{pd}_C M \leq 1$ if and only if $D(B \otimes_C DM) \cong M$.

Lemma 2.21. [22, Lemma 4.4]. *Let C be an algebra of global dimension 2 and M a C -module.*

- (a) $\text{pd}_C N = 2$ for all nonzero $N \in \text{add}(M \otimes_C E)$.
- (b) $\text{id}_C N = 2$ for all nonzero $N \in \text{add}(D(E \otimes_C DM))$.

We end this section with a lemma which tells us what the projective cover of a projective C -module is in $\text{mod } B$.

Lemma 2.22. [2, Lemma 2.7] *Let C be an algebra of global dimension at most 2 and $B = C \ltimes E$. Suppose P is a projective C -module. Then the induced module, $P \otimes_C B$, is a projective cover of P in $\text{mod } B$.*

We also have the following important fact.

Lemma 2.23. [8, Corollary 1.2]. $\tau_C M$ and $\tau_B(M \otimes_C B)$ are submodules of $\tau_B M$.

3 Homological Dimensions

In this section let C be an algebra of global dimension 2, $E = \text{Ext}_C^2(DC, C)$, and $B = C \ltimes E$ be the relation extension. We investigate what happens to the projective dimension of a C -module M when viewed as a B -module. In the special case when C is a tilted algebra and B is the corresponding cluster-tilted algebra, we provide a complete classification. First, we prove a lemma which provides a useful criteria for a C -module to have projective or injective dimension at most 1 in an algebra of global dimension 2.

Lemma 3.1. *Let M be a C -module. Then,*

- (a) $\text{pd}_C M \leq 1$ if and only if $\text{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, M) = 0$.
- (b) $\text{id}_C M \leq 1$ if and only if $\text{Hom}_C(M, \tau_C \Omega_C DC) = 0$.

Proof. We prove (a) with the proof of (b) being similar. Assume $\text{pd}_C M \leq 1$. Consider the short exact sequence

$$0 \rightarrow C \rightarrow I_0 \rightarrow \Omega_C^{-1} C \rightarrow 0$$

where I_0 is an injective envelope of C . Apply $\text{Hom}_C(M, -)$ to obtain an exact sequence

$$\text{Ext}_C^1(M, I_0) \rightarrow \text{Ext}_C^1(M, \Omega_C^{-1} C) \rightarrow \text{Ext}_C^2(M, C).$$

Now, $\text{Ext}_C^1(M, I_0) = 0$ because I_0 is injective and $\text{Ext}_C^2(M, C) = 0$ because $\text{pd}_C M \leq 1$. Since the sequence is exact, $\text{Ext}_C^1(M, \Omega_C^{-1}C) = 0$. By the Auslander-Reiten formulas, $D\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) \cong \text{Ext}_C^1(M, \Omega_C^{-1}C)$. Thus,

$$0 = \text{Ext}_C^1(M, \Omega_C^{-1}C) \cong D\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M).$$

Conversely, assume $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) = 0$. Then we have

$$D\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) \cong \text{Ext}_C^1(M, \Omega_C^{-1}C) = 0$$

by the Auslander-Reiten formulas. We then have $\text{Ext}_C^2(M, C) \cong \text{Ext}_C^1(M, \Omega_C^{-1}C) = 0$. Since C has global dimension equal to 2, this implies $\text{pd}_C M \leq 1$. \square

We begin with the case where M is a projective C -module.

Proposition 3.2. *Let M be a projective C -module. Then $\text{pd}_B M = 0$ if and only if $\text{id}_C M \leq 1$.*

Proof. Assume $\text{pd}_B M = 0$. By Proposition 2.10 we have a short exact sequence

$$0 \rightarrow \tau_C^{-1}\Omega_C^{-1}M \rightarrow M \otimes_C B \rightarrow M \rightarrow 0$$

where $M \otimes_C B$ is a projective cover by Lemma 2.11. This implies $M \otimes_C B \cong M$ and $\tau_C^{-1}\Omega_C^{-1}M = 0$. By Proposition 2.20, we conclude $\text{id}_C M \leq 1$.

Conversely, assume $\text{id}_C M \leq 1$. Then Proposition 2.20 implies $M \otimes_C B \cong M$ and we conclude M is a projective B -module. \square

The case where the projective dimension of M is equal to 2 holds in a more general setting which we explicitly state.

Proposition 3.3. *Let C be an algebra of global dimension 2 with B a split extension by a nilpotent bimodule E . If M is a C -module with $\text{pd}_C M = 2$, then $\text{pd}_B M \geq 2$.*

Proof. By Lemma 2.13, we have $\text{pd}_B(M \otimes_C B) \geq 2$. This implies the existence of a non-zero morphism $f: DB \rightarrow \tau_B(M \otimes_C B)$ by Lemma 2.16. By Lemma 2.23, we have an injective morphism $i: \tau_B(M \otimes_C B) \rightarrow \tau_B M$. Thus, there is a non-zero morphism $i \circ f: DB \rightarrow \tau_B M$. By Lemma 2.16 again, we have $\text{pd}_B M \geq 1$. \square

The case where the projective dimension of M is equal to 1 is the most restrictive.

Proposition 3.4. *Let M be a C -module with $\text{pd}_C M = 1$ and a minimal projective resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ in $\text{mod } C$. Then $\text{id}_C M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_1 \cong \tau_C^{-1}\Omega_C^{-1}P_0$ if and only if $\text{pd}_B M = 1$.*

Proof. Assume $\text{id}_C M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_1 \cong \tau_C^{-1}\Omega_C^{-1}P_0$. Since $\text{id}_C M \leq 1$, by Proposition 2.20, we have $M \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1}M = 0$ and $M \otimes_C B \cong M$. Using Lemma 2.13, we need to show $\text{Hom}_C(DE, \tau_C M) = 0$. Apply $- \otimes_C E$ to the minimal projective resolution of M to obtain the exact sequence

$$\text{Tor}_1^C(P_1, E) \rightarrow \text{Tor}_1^C(M, E) \rightarrow P_1 \otimes_C E \rightarrow P_0 \otimes_C E \rightarrow M \otimes_C E \rightarrow 0. \quad (1)$$

Now, $\text{Tor}_1^C(P_1, E) = 0$ because P_1 is projective and we showed $M \otimes_C E = 0$. Also, Proposition 2.20 says $P_1 \otimes_C E \cong \tau_C^{-1} \Omega_C^{-1} P_1 \cong \tau_C^{-1} \Omega_C^{-1} P_0 \cong P_0 \otimes_C E$. Since (1) is exact, we know $\text{Tor}_1^C(M, E) = 0$. By Proposition 2.15 and the Auslander-Reiten formulas, we have

$$0 = \text{Tor}_1^C(M, E) \cong D\text{Ext}_C^1(M, DE) \cong \overline{\text{Hom}}_C(DE, \tau_C M).$$

Since $\text{pd}_C M = 1$ by assumption, we may use Lemma 2.16 and the Auslander-Reiten formulas to say

$$0 = \overline{\text{Hom}}_C(DE, \tau_C M) \cong \text{Hom}_C(DE, \tau_C M).$$

Conversely, assume $\text{pd}_B M = 1$. If $\text{pd}_B(M \otimes_C B) > 1$ then we have a non-zero composition of morphisms, $DB \rightarrow \tau_B(M \otimes_C B) \rightarrow \tau_B M$, guaranteed by Lemma 2.16 and Lemma 2.23. By Lemma 2.16, this contradicts $\text{pd}_B M = 1$. Thus, $\text{pd}_B(M \otimes_C B) = 1$ and Proposition 2.15, Lemma 2.13, the Auslander-Reiten formulas, and Lemma 2.16 imply

$$0 = \text{Hom}_C(DE, \tau_C M) \cong D\text{Ext}_C^1(M, DE) \cong \text{Tor}_1^C(M, E).$$

Next, consider the short exact sequence of Propositions 2.10 and Proposition 2.19

$$0 \rightarrow \tau_C^{-1} \Omega_C^{-1} M \rightarrow M \otimes_C B \rightarrow M \rightarrow 0$$

in $\text{mod } B$. Since $\text{pd}_B(M \otimes_C B)$ and $\text{pd}_B M$ are equal to 1, we know Lemma 2.14 implies $\text{pd}_B(\tau_C^{-1} \Omega_C^{-1} M) \leq 1$. By Lemma 2.21, we know $\text{pd}_C(\tau_C^{-1} \Omega_C^{-1} M) = 2$ or $\tau_C^{-1} \Omega_C^{-1} M = 0$. However, Proposition 3.3 implies $\text{pd}_B(\tau_C^{-1} \Omega_C^{-1} M) \geq 2$. Thus, $\tau_C^{-1} \Omega_C^{-1} M = 0$ and $M \otimes_C B \cong M$. Returning to sequence (1), since $M \otimes_C B \cong M$ we have $M \otimes_C E = 0$. Also, we have shown that $\text{Tor}_1^C(M, E) = 0$. Since the sequence is exact, we have $P_1 \otimes_C E \cong P_0 \otimes_C E$ and Proposition 2.19 implies $\tau_C^{-1} \Omega_C^{-1} P_1 \cong \tau_C^{-1} \Omega_C^{-1} P_0$. Finally, since $M \otimes_C E = 0$, Proposition 2.20 tells us that $\text{id}_C M \leq 1$. \square

If M is a C -module which satisfies the conditions of Proposition 3.4, then the following corollary tells us what a minimum projective resolution is in $\text{mod } B$.

Corollary 3.5. *Let M be a C -module with minimal projective resolution*

$$0 \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0.$$

If $\text{pd}_B M = 1$, then $0 \rightarrow P_1 \otimes_C B \xrightarrow{f_1 \otimes_C 1_B} P_0 \otimes_C B \xrightarrow{f_0 \otimes_C 1_B} M \rightarrow 0$ is a minimal projective resolution in $\text{mod } B$.

Proof. By Lemma 2.12, we know that $P_1 \otimes_C B \xrightarrow{f_1 \otimes_C 1_B} P_0 \otimes_C B \xrightarrow{f_0 \otimes_C 1_B} M \otimes_C B \rightarrow 0$ is a minimal projective presentation of $M \otimes_C B$ in $\text{mod } B$. By Proposition 3.4, we know $\text{id}_C M \leq 1$. By Proposition 2.20 we have $M \otimes_C B \cong M$ and our statement follows. \square

In the situation where C is an algebra of global dimension 2 and B is a split extension by a nilpotent bimodule E , we prove that the global dimension of B is strictly greater than the global dimension of C . We need a lemma.

Lemma 3.6. *Let M be a projective C -module such that $\text{id}_C M = 2$. Then*

$$\text{pd}_B M = \text{pd}_B(\tau_C^{-1} \Omega_C^{-1} M) + 1 \geq 3.$$

Proof. Consider the short exact sequence $0 \rightarrow \tau_C^{-1}\Omega_C^{-1}M \rightarrow M \otimes_C B \rightarrow M \rightarrow 0$ guaranteed by Proposition 2.10 and Proposition 2.19. We have $\text{pd}_B(M \otimes_C B) = 0$ and $\text{pd}_B(\tau_C^{-1}\Omega_C^{-1}M) \geq 2$ by Proposition 3.3. Our statement then follows from Lemma 2.14. \square

Corollary 3.7. *Let C be an algebra of global dimension 2 and B a split extension by a nilpotent bimodule E . Then $\text{gl.dim. } B > \text{gl.dim. } C$.*

Proof. This follows immediately from Lemma 3.6. \square

We conclude this section with a complete classification of the projective dimension of a C -module when viewed as a B -module in the special case C is tilted and B is the corresponding cluster-tilted algebra.

Theorem 3.8. *Let C be a tilted algebra, $E = \text{Ext}_C^2(DC, C)$, and $B = C \ltimes E$ the corresponding cluster-tilted algebra.*

- (a) *If $\text{pd}_C M = 0$, then $\text{pd}_B M = 0$ if and only if $\text{id}_C M \leq 1$. Otherwise, $\text{pd}_B M = \infty$.*
- (b) *If $\text{pd}_C M = 2$, then $\text{pd}_B M = \infty$.*
- (c) *Let $\text{pd}_C M = 1$ with minimal projective resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Then $\text{pd}_B M = 1$ if and only if $\text{id}_C M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_0 \cong \tau_C^{-1}\Omega_C^{-1}P_1$. Otherwise, $\text{pd}_B M = \infty$.*

Proof. Part (a) follows from Proposition 3.2. If the conditions for M are not met, then Lemma 3.6 and the 1-Gorenstein property of a cluster-tilted algebra, (Theorem 2.5), shows $\text{pd}_B M = \infty$. Part (b) follows from Proposition 3.3 and the 1-Gorenstein property. Finally, part (c) follows from Proposition 3.4 and the 1-Gorenstein property. \square

For an illustration of this theorem, see Examples 5.1, 5.2, and 5.3 in section 5.

4 Extensions

In this section, we study C -modules which have no self-extension, i.e., $\text{Ext}_C^1(M, M) = 0$. These modules are typically referred to as rigid modules. We investigate under what conditions does a rigid C -module remain a rigid B -module. Unless otherwise stated, we assume that C is an algebra of global dimension 2 and $B = C \ltimes E$ is a split extension by a nilpotent bimodule E . To prove our main result we first need an easy lemma. We recall from Lemma 2.22 that if P is a projective C -module, then $P \otimes_C B$ is a projective cover of P in $\text{mod } B$.

Lemma 4.1. *Let M be a C -module with $f : P_0 \rightarrow M$ a projective cover in $\text{mod } C$. Suppose $g : P_0 \otimes_C B \rightarrow P_0$ is a projective cover of P_0 in $\text{mod } B$. Then $f \circ g : P_0 \otimes_C B \rightarrow M$ is a projective cover of M in $\text{mod } B$.*

Proof. Clearly, $f \circ g$ is surjective. Thus, we need to show $\ker f \circ g$ is superfluous. This follows easily from Corollary 2.18 since f and g are both minimal. \square

Theorem 4.2. *Let M be a rigid C -module with a projective cover $P_0 \rightarrow M$ and an injective envelope $M \rightarrow I_0$ in $\text{mod } C$.*

- (a) *If $\text{Hom}_C(\tau_C^{-1}\Omega^{-1}P_0, M) = 0$, then M is a rigid B -module.*
- (b) *If $\text{Hom}_C(M, \tau_C\Omega I_0) = 0$, then M is a rigid B -module.*

Proof. We prove case (a) with case (b) being dual. In $\text{mod } B$, consider the following short exact sequence of M

$$0 \rightarrow \Omega_B^1 M \xrightarrow{f} P_0 \otimes_C B \rightarrow M \rightarrow 0.$$

Apply $\text{Hom}_B(-, M)$ to obtain

$$0 \rightarrow \text{Hom}_B(M, M) \rightarrow \text{Hom}_B(P_0 \otimes_C B, M) \xrightarrow{\bar{f}} \text{Hom}_B(\Omega_B^1 M, M) \rightarrow \text{Ext}_B^1(M, M) \rightarrow 0. \quad (1)$$

Since (1) is exact, we need to show that \bar{f} is surjective. This will imply that $\text{Ext}_B^1(M, M) = 0$. In $\text{mod } C$, consider the sequence

$$0 \rightarrow \Omega_C^1 M \xrightarrow{g} P_0 \xrightarrow{a} M \rightarrow 0.$$

Apply $\text{Hom}_C(-, M)$ to obtain

$$0 \rightarrow \text{Hom}_C(M, M) \rightarrow \text{Hom}_C(P_0, M) \xrightarrow{\bar{g}} \text{Hom}_C(\Omega_C^1 M, M) \rightarrow \text{Ext}_C^1(M, M). \quad (2)$$

Since M is a rigid C -module by assumption and (2) is exact, we have \bar{g} is surjective. Next, in $\text{mod } B$, consider the following commutative diagram guaranteed by Lemma 4.1 and the universal property of the kernel.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_B^1 M & \xrightarrow{f} & P_0 \otimes_C B & \xrightarrow{a \circ w} & M \longrightarrow 0 \\ & & \downarrow z & & \downarrow w & & \downarrow id \\ 0 & \longrightarrow & \Omega_C^1 M & \xrightarrow{g} & P_0 & \xrightarrow{a} & M \longrightarrow 0. \end{array} \quad (3)$$

Here, id is the identity map, w is a projective cover of P_0 , and z is induced by the universal property of the kernel. By the Snake Lemma, we know $\ker z \cong \ker w$. Thus, Proposition 2.10 and Proposition 2.19 implies that $\ker z \cong \ker w \cong \tau_C^{-1}\Omega_C^{-1}P_0$. Thus, we have an exact sequence

$$0 \rightarrow \tau_C^{-1}\Omega_C^{-1}P_0 \xrightarrow{i} \Omega_B^1 M \xrightarrow{\bar{z}} \Omega_C^1 M \rightarrow \text{coker } z \rightarrow 0.$$

Since the morphism w is surjective and id is clearly injective, we may use the Snake Lemma again to say that $\text{coker } z = 0$. Apply $\text{Hom}_B(-, M)$ to obtain an exact sequence

$$0 \rightarrow \text{Hom}_B(\Omega_C^1 M, M) \xrightarrow{\bar{z}} \text{Hom}_B(\Omega_B^1 M, M) \rightarrow \text{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M). \quad (1)$$

Since $\text{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ by assumption, we have that \bar{z} is an isomorphism. To show \bar{f} is surjective, let $h \in \text{Hom}_B(\Omega_B^1 M, M)$. Since \bar{z} is an isomorphism, we know there exists a morphism $j \in \text{Hom}_B(\Omega_C^1 M, M)$ such that $h = j \circ z$.

$$\begin{array}{ccc} \Omega_B^1 M & & \\ \downarrow z & \searrow h = j \circ z & \\ \Omega_C^1 M & \xrightarrow{j} & M \end{array}$$

Since \bar{g} is surjective, there exists a morphism $l \in \text{Hom}_B(P_0, M)$ such that $j = l \circ g$.

$$\begin{array}{ccc} \Omega_C^1 M & \xrightarrow{j = l \circ g} & M \\ \downarrow g & \nearrow l & \\ P_0 & & \end{array}$$

Thus, we have $h = l \circ g \circ z$.

$$\begin{array}{ccc} \Omega_B^1 M & & \\ \downarrow z & \searrow h = l \circ g \circ z & \\ \Omega_C^1 M & & \\ \downarrow g & & \\ P_0 & \xrightarrow{l} & M \end{array}$$

From our commutative diagram (3), we know $g \circ z = w \circ f$. Thus, we have the following commutative diagram.

$$\begin{array}{ccc} \Omega_B^1 M & & \\ \downarrow f & \searrow h = l \circ w \circ f & \\ P_0 \otimes_C B & & \\ \downarrow w & & \\ P_0 & \xrightarrow{l} & M \end{array}$$

This gives $h = l \circ w \circ f$ and we conclude that \bar{f} is surjective. \square

For an illustration of this theorem, see Examples 5.4 and 5.5 in section 5.

4.1 Corollaries

We now examine several corollaries of our main result. For the first corollary, we say M is a *partial cotilting module* if $\text{id}_C M \leq 1$ and $\text{Ext}_C^1(M, M) = 0$ and *cotilting* if the number of pairwise, non-isomorphic, indecomposable summands of M equals the number of isomorphism classes of simple C -modules.

Corollary 4.3. *If M is a partial tilting or cotilting C -module, then M is a rigid B -module.*

Proof. We assume M is a partial tilting module. The proof for the case M is a partial cotilting module is dual. Since $\text{pd}_C M \leq 1$, we have that Lemma 3.1 implies $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) = 0$. The statement now follows from Theorem 4.2. \square

The next result holds in the specific case where C is tilted and B is cluster-tilted.

Corollary 4.4. *Let C be a tilted algebra with B the corresponding cluster-tilted algebra. Suppose M is an indecomposable, rigid C -module. Then M is a rigid B -module.*

Proof. Let M be an indecomposable, rigid C -module. By Proposition 2.4(b), we have that $\text{pd}_C M \leq 1$ or $\text{id}_C M \leq 1$. Since M is rigid, we have M is partial tilting or partial cotilting. By Corollary 4.3, our statement follows. \square

We now state the converse to Theorem 4.2. We note that if M is a C -module which is rigid as a B -module, then M is trivially a rigid C -module.

Proposition 4.5. *Assume C is an algebra of global dimension 2. Let M be a C -module with a projective cover $g: P_0 \rightarrow M$ and an injective envelope $h: M \rightarrow I_0$ in $\text{mod } C$. Suppose M is a rigid B -module.*

- (a) *If $\text{Ext}_B^1(P_0, M) = 0$, then $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$.*
- (b) *If $\text{Ext}_B^1(M, I_0) = 0$, then $\text{Hom}_C(M, \tau_C\Omega_C I_0) = 0$.*

Proof. We prove case (a) with case (b) being dual. Consider the following sequence in $\text{mod } B$ guaranteed by Proposition 2.10 and Proposition 2.19.

$$0 \rightarrow \tau_C^{-1}\Omega_C^{-1}P_0 \xrightarrow{f} P_0 \otimes_C B \rightarrow P_0 \rightarrow 0.$$

Apply $\text{Hom}_B(-, M)$ to obtain

$$0 \rightarrow \text{Hom}_B(P_0, M) \rightarrow \text{Hom}_B(P_0 \otimes_C B, M) \xrightarrow{\bar{f}} \text{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) \rightarrow \text{Ext}_B^1(P_0, M).$$

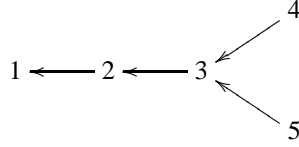
Since the sequence is exact and $\text{Ext}_B^1(P_0, M) = 0$ by assumption, we have that \bar{f} is surjective. This implies that any morphism of B -modules, $j: \tau_C^{-1}\Omega_C^{-1}P_0 \rightarrow M$, factors through the projective B -module $P_0 \otimes_C B$. Since $g: P_0 \rightarrow M$ is a surjective morphism, there exists a morphism $k: \tau_C^{-1}\Omega_C^{-1}P_0 \rightarrow P_0$ such that $j = g \circ k$.

$$\begin{array}{ccc} & \tau_C^{-1}\Omega_C^{-1}P_0 & \\ k \swarrow & \downarrow j = g \circ k & \\ P_0 & \xrightarrow{g} & M \end{array}$$

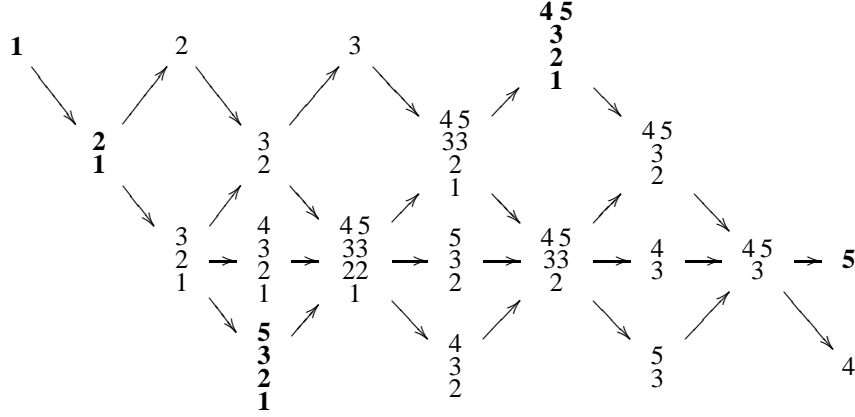
But $\text{pd}_C P_0 = 0$ and Lemma 3.1 implies $\text{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, P_0) = 0$. Thus k must be the 0 morphism. This forces j to also be the 0 morphism. Since j was arbitrary we conclude that $\text{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ which further implies $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ by restriction of scalars. \square

5 Examples

In this section we illustrate our main results with several examples. We will use the following throughout this section. Let A be the path algebra of the following quiver:



Since A is a hereditary algebra, we may construct a tilted algebra. To do this, we need an A -module which is tilting. Consider the Auslander-Reiten quiver of A which is given by:



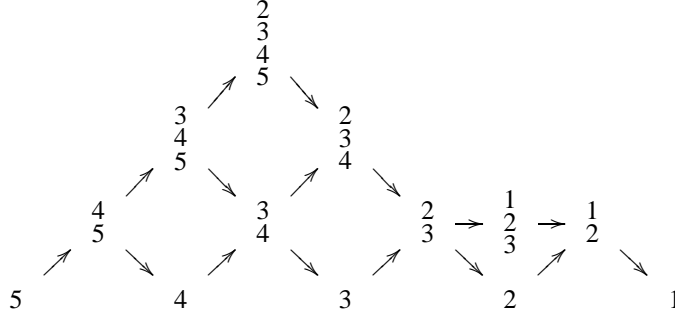
Let T be the tilting A -module

$$T = 5 \oplus \begin{pmatrix} 4 & 5 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \end{pmatrix} \oplus 1$$

The corresponding tilted algebra $C = \text{End}_A T$ is given by the bound quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5 \quad \alpha\beta\gamma = 0$$

Then, the Auslander-Reiten quiver of C is given by:

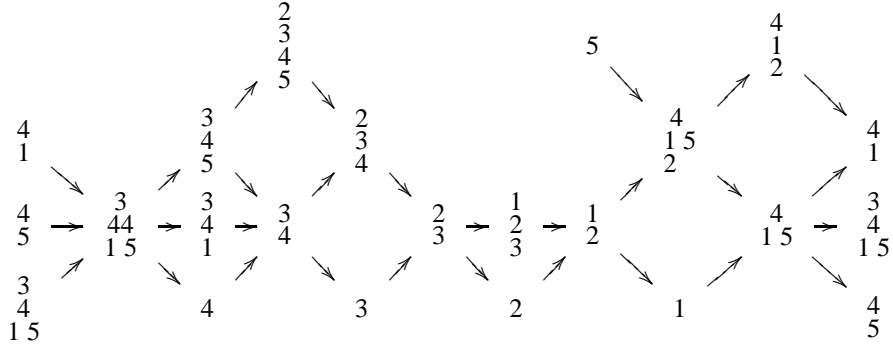


The corresponding cluster-tilted algebra $B = C \ltimes \text{Ext}_C^2(DC, C)$ is given by the bound quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5 \quad \alpha\beta\gamma = \beta\gamma\delta = \gamma\delta\alpha = \delta\alpha\beta = 0$$

δ

Then, the Auslander-Reiten quiver of B is given by:



We wish to illustrate Theorem 3.8 with an example for each case. We will use Lemma 3.1 frequently so we note that

$$\tau_C^{-1}\Omega_C^{-1}C = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \oplus 1, \quad \tau_C\Omega_C(DC) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \oplus 4.$$

Example 5.1. We'll start with the projective dimension equal to 2. In $\text{mod } C$, consider the module $M = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Since $\text{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) \neq 0$, Lemma 3.1 says $\text{pd}_C M = 2$. Thus, Theorem 3.8 says $\text{pd}_B M = \infty$ and we have the following projective resolution in $\text{mod } B$

$$\cdots \rightarrow \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \\ 1 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow 0.$$

Example 5.2. Next, let's examine the projective case, i.e., projective dimension equal to 0. In $\text{mod } C$, consider the module $M = 5$. Then M is the projective C -module at vertex 5. Since $\text{Hom}_C(M, \tau_C \Omega_C(DC)) = 0$, Lemma 3.1 says $\text{id}_C M \leq 1$. Thus, Theorem 3.8 says $\text{pd}_B M = 0$. Now, consider $N = \frac{4}{5}$ in $\text{mod } C$. Then N is a projective C -module. Now, we have that $\text{Hom}_C(N, \tau_C \Omega_C(DC)) \neq 0$. Thus, Lemma 3.1 says $\text{id}_C N = 2$. Finally, Theorem 3.8 states $N = \frac{4}{5}$ is not a projective B -module and $\text{pd}_B N = \infty$ with the following projective resolution in $\text{mod } B$

$$\cdots \rightarrow \frac{3}{4} \rightarrow \frac{1}{2} \rightarrow \frac{4}{15} \rightarrow \frac{4}{5} \rightarrow 0.$$

Example 5.3. Finally, let's examine the case where the projective dimension is equal to 1. Consider the C -module $M = \frac{3}{4}$. Since $\text{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, M) = 0$, Lemma 3.1 says $\text{pd}_C M \leq 1$ with projective resolution

$$0 \rightarrow 5 \rightarrow \frac{3}{4} \rightarrow \frac{3}{4} \rightarrow 0.$$

Denote $P_1 = 5$ and $P_0 = \frac{3}{4}$. Since $\text{Hom}_C(M, \tau_C \Omega_C(DC)) \neq 0$, Lemma 3.1 says $\text{id}_C M = 2$. Also, note that $\tau_C^{-1} \Omega_C^{-1} P_1 \not\cong \tau_C^{-1} \Omega_C^{-1} P_0$ because $\tau_C^{-1} \Omega_C^{-1} P_1 = 0$ while $\tau_C^{-1} \Omega_C^{-1} P_0 = \frac{1}{2}$. Thus, Theorem 3.8 says $\text{pd}_B M = \infty$ and we have the following projective resolution in $\text{mod } B$

$$\cdots \rightarrow \frac{2}{3} \rightarrow \frac{3}{4} \rightarrow 5 \oplus \frac{1}{2} \rightarrow \frac{3}{4} \rightarrow \frac{3}{4} \rightarrow 0.$$

Next, consider the C -module $N = \frac{2}{3}$. Since $\text{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, N) = 0$, Lemma 3.1 says that $\text{pd}_C N = 1$ with minimal projective resolution

$$0 \rightarrow 5 \rightarrow \frac{2}{3} \rightarrow \frac{2}{3} \rightarrow 0.$$

Denote $P'_1 = 5$ and $P'_0 = \frac{2}{3}$. Since $\text{Hom}_C(N, \tau_C \Omega_C(DC)) = 0$, Lemma 3.1 says

that $\text{id}_C N \leq 1$. Also, note that $\tau_C^{-1} \Omega_C^{-1} P'_1 \cong \tau_C^{-1} \Omega_C^{-1} P'_0 = 0$. Thus, Theorem 3.8 says $\text{pd}_B N = 1$ and Corollary 3.5 implies the minimal projective resolution in $\text{mod } C$ is the same as the minimal projective resolution in $\text{mod } B$.

We will illustrate Theorem 4.2 with two examples.

Example 5.4. Consider the C -module $M = \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$. To use Theorem 4.2 we need several preliminary calculations. We have a projective cover and an injective envelope in $\text{mod } C$

$$f: \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \rightarrow M, \quad g: M \rightarrow \begin{smallmatrix} 2 \\ 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}.$$

Let us denote $\begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ by P_0 and $\begin{smallmatrix} 2 \\ 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ by I_0 . Then we have

$$\tau_C^{-1} \Omega_C^{-1} P_0 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \quad \tau_C \Omega_C I_0 = \begin{smallmatrix} 4 \end{smallmatrix}.$$

It is easily seen that $\text{Ext}_C^1(M, M) = 0$ but $\text{Ext}_B^1(M, M) \neq 0$ with self-extension

$$0 \rightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \rightarrow \begin{smallmatrix} 4 \\ 1 \\ 5 \\ 2 \end{smallmatrix} \rightarrow \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \rightarrow 0$$

Note that $\text{Hom}_C(\tau_C^{-1} \Omega_C^{-1} P_0, M) \neq 0$ and $\text{Hom}_C(M, \tau_C \Omega_C I_0) \neq 0$ in accordance with Theorem 4.2.

Example 5.5. Consider the C -module $N = \begin{smallmatrix} 5 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}$. We have a projective cover and an injective envelope in $\text{mod } C$

$$f: \begin{smallmatrix} 5 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \rightarrow M, \quad g: M \rightarrow \begin{smallmatrix} 2 \\ 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}.$$

Denote $\begin{smallmatrix} 5 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ by P_0 and $\begin{smallmatrix} 2 \\ 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$ by I_0 . Then we have

$$\tau_C^{-1} \Omega_C^{-1} P_0 = \begin{smallmatrix} 1 \end{smallmatrix}, \quad \tau_C \Omega_C I_0 = \begin{smallmatrix} 0 \end{smallmatrix}.$$

Now, we have $\text{Ext}_C^1(M, M) = 0$ and $\text{Ext}_B^1(M, M) = 0$. Note, $\text{Hom}_C(\tau_C^{-1} \Omega_C^{-1} P_0, M) = 0$ and $\text{Hom}_C(M, \tau_C \Omega_C I_0) = 0$ in accordance with Theorem 4.2.

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References

- [1] C. Amoit, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier* **59**, (2009), no. 6, 2525–2590.

- [2] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras as trivial extensions, *Bull. Lond. Math. Soc.* **40** (2008), 151–162.
- [3] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras and slices, *J. of Algebra* **319** (2008), 3464–3479.
- [4] I. Assem, T. Brüstle and R. Schiffler, On the Galois covering of a cluster-tilted algebra, *J. Pure Appl. Alg.* **213** (7) (2009), 1450–1463.
- [5] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras without clusters *J. Algebra* **324**, (2010), 2475–2502.
- [6] I. Assem and N. Marmaridis, Tilting modules and split-by-nilpotent extensions, *Comm. Algebra* **26** (1998), 1547–1555.
- [7] I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, Cambridge University Press, 2006.
- [8] I. Assem and D. Zacharia, Full embeddings of almost split sequences over split-by-nilpotent extensions, *Coll. Math.* **81**, (1) (1999), 21–31.
- [9] M. Barot, E. Fernandez, I. Pratti, M. I. Platzeck and S. Trepode, From iterated tilted to cluster-tilted algebras, *Adv. Math.* **223** (2010), no. 4, 1468–1494.
- [10] L. Beaudet, T. Brüstle and G. Todorov, Projective dimension of modules over cluster-tilted algebras, *Algebr. and Represent. Theory* **17** (2014), no. 6, 1797–1807.
- [11] M. A. Bertani-Økland, S. Oppermann and A. Wrålsen, Constructing tilted algebras from cluster-tilted algebras, *J. Algebra* **323** (2010), no. 9, 2408–2428.
- [12] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204** (2006), no. 2, 572–618.
- [13] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras, *Trans. Amer. Math. Soc.* **359** (2007), no. 1, 323–332.
- [14] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras of finite representation type, *J. Algebra* **306** (2006), no. 2, 412–431.
- [15] A. B. Buan, R. Marsh and I. Reiten, Cluster mutation via quiver representations, *Comment. Math. Helv.* **83** (2008), no. 1, 143–177.
- [16] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* **358** (2006), no. 4, 359–376.
- [17] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations and cluster tilted algebras, *Algebr. and Represent. Theory* **9**, (2006), no. 4, 359–376.
- [18] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.

- [19] B. Keller, On triangulated orbit categories, *Documenta Math.* **10** (2005), 551–581.
- [20] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* **211** (2007), no. 1, 123–151.
- [21] P. G. Plamondon, Cluster algebras via cluster categories with infinite-dimensional morphism spaces, *Compos. Math.* **147** (2011), no. 6, 1921–1954.
- [22] R. Schiffler and K. Serhiyenko, Induced and coinduced modules in cluster-tilted algebras, preprint, **arXiv : 1410.1732**.

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